E9 203 Compressed Sensing & Sparse Signal Processing

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Recap:

1. Gelfand m-width:
$$d^m(K,X) = \inf_{A \in \mathbb{R}^{m\times N}} \sup_{x \in K \cap \mathcal{N}(A)} \|x\|$$

$K = B_1^N$, $X = \ell_p^N$

2. Compressive m-width:
$$E^m(K,X) = \inf \left\{ \sup_{x\in K} \|x - \Delta(Ax)\|, \; A: X \to \mathbb{R}^m \text{ linear}, \; \Delta: \mathbb{R}^m \to X \right\}.$$

Thm. 10.5 For $1 < p \le 2$ and $m < N$, $\exists$ consts. $c_1, c_2 > 0$ dep. only on p s.t.
$$\boxed{c_1 \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \le d^m(B_1^N, \ell_p^N) \le c_2 \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}.}$$

Consequence: (Cor. 10.6)
$$c_1' \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \le E^m(B_1^N, \ell_p^N) \le c_2' \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}.$$

Consequence of the lower bound (Prop. 10.7)

If $\exists\, A \in \mathbb{R}^{m\times N}$ & $\Delta: \mathbb{R}^m \to \mathbb{R}^N$ s.t. $\forall x \in \mathbb{R}^N$
$$\|x - \Delta(Ax)\|_p \le \frac{C}{s^{1-1/p}} \sigma_s(x)_1, \quad 1 < p \le 2,$$
then, for consts $c_1, c_2 > 0$ dep. only on C,
$$m \ge c_1 s \ln\left(\frac{eN}{s}\right), \text{ provided } s > c_2.$$

Proof of thm 10.5:

Upper bound: Uses Thm. 6.12.

Lower bound:

Thm. 10.10: There is a const. $c > 0$ s.t. for $1 < p \le \infty$ and $m < N$,
$$d^m(B_1^N, \ell_p^N) \ge c \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-1/p}.$$

Proof: Relies on two results:

Lemma 10.12 Given integers $s < N$, $\exists\, n \ge \left(\frac{N}{4s}\right)^{s/2}$ subsets $S_1, \dots, S_n$ of $[N]$ s.t. each S_j has cardinality s and
$$|S_i \cap S_j| < \frac{s}{2} \quad \forall\, i \ne j.$$

Thm. 10.11 Given $A \in \mathbb{R}^{m\times N}$, if every $2s$-sparse $x \in \mathbb{R}^N$ is a minimizer of $\|z\|_1$ s.t. $Az = Ax$, then
$$m \ge c_1 s \ln\left(\frac{N}{c_2 s}\right),$$
where $c_1 = \frac{1}{\ln 9}$ and $c_2 = 4$.

Proof of Thm. 10.10

Will show that, with $c' \triangleq \frac{2}{1 + 4\ln 9} = \frac{2c_1}{4 + c_1}$,
$$d^m(B_1^N, \ell_p^N) \ge \frac{\mu^{1-1/p}}{2^{2-1/p}}, \text{ where } \boxed{\mu \triangleq \min\left\{1, \frac{c' \ln\left(\frac{eN}{m}\right)}{m}\right\}}$$

Then the result follows with $c = \frac{\min\{1, c'\}^{1-1/p}}{2^{2-1/p}} \ge \frac{\min\{1, c'\}}{4}$
$$\left(d^m(B_1^N, \ell_p^N) \ge c \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-1/p}. \right)$$

Proof by contradiction.

Suppose $d^m(B_1^N, \ell_p^N) < \frac{\mu^{1-1/p}}{2^{2-1/p}}$.

Recall: $d^m(B_1^N, \ell_p^N) = \inf_{A \in \mathbb{R}^{m\times N}} \sup_{x \in B_1^N \cap \mathcal{N}(A)} \|x\|_p$.

$$d^m(B_1^N, \ell_p^N) < \frac{\mu^{1-1/p}}{2^{2-1/p}} \Rightarrow \exists\, A \in \mathbb{R}^{m\times N} \text{ s.t.}$$

$\forall\, v \in \mathcal{N}(A)$, $v \ne 0$,
$$\boxed{\|v\|_p < \frac{\mu^{1-1/p}}{2^{2-1/p}} \|v\|_1}.$$

Also, define integer $s \ge 1$ by $s \triangleq \left\lfloor \frac{1}{\mu} \right\rfloor$, so that
$$\frac{1}{2\mu} < s \le \frac{1}{\mu}.$$

Then, $\forall\, v \in \mathcal{N}(A) \setminus \{0\}$,
$$\|v\|_p < \frac{1}{2}\left(\frac{\mu}{2}\right)^{1-1/p} \|v\|_1 \le \frac{1}{2}\left(\frac{1}{2s}\right)^{1-\frac{1}{p}} \|v\|_1 \quad \text{--- } (*1)$$

Recall that $\|v\|_1 \le N^{1-\frac{1}{p}} \|v\|_p$ for any $v \in \mathbb{R}^N$, any $p \ge 1$.

$\Rightarrow$ For a $v \in B_1^N \cap \mathcal{N}(A)$,
$$1 = \|v\|_1 \le N^{1-\frac{1}{p}} \|v\|_p < \frac{N^{1-\frac{1}{p}}}{2}\left(\frac{1}{2s}\right)^{1-\frac{1}{p}} \underbrace{\|v\|_1}_{=1}$$
$$\Rightarrow 1 < \frac{1}{2}\left(\frac{N}{2s}\right)^{1-\frac{1}{p}} \Rightarrow 2s < N.$$

Then, for $S \subset [N]$ with $|S| \le 2s$ and for $v \in \mathcal{N}(A) \setminus \{0\}$,
$$\|v_S\|_1 \le (2s)^{1-\frac{1}{p}} \|v_S\|_p \quad (\text{since } |S| \le 2s)$$
$$\le (2s)^{1-1/p} \|v\|_p < (2s)^{1-\frac{1}{p}} \cdot \frac{1}{2}\left(\frac{1}{2s}\right)^{1-1/p} \cdot \|v\|_1$$
$$\Rightarrow \boxed{\|v_S\|_1 < \frac{1}{2}\|v\|_1}.$$

NSP: $v \in \mathcal{N}(A)\setminus\{0\}$,
$\|v_S\|_1 < \|v_{S^c}\|_1$
$2\|v_S\|_1 < \|v\|_1$

Thus, A satisfies the NSP of order $2s$.

$\Rightarrow$ From Thm. 4.5 : every $2s$-sparse vec. [illegible]
uniquely recovered from $y = Ax$ via ℓ_1-min.

Thm. 10.11 now $\Rightarrow$

$$\underline{m \geq c_1 s \ln\left(\frac{N}{c_2 s}\right)}, \quad c_1 = \frac{1}{\ln 9}, \quad c_2 = 4.$$

Thm. 2.13 [4 equivalent props. to " Every $\underline{s\text{-sparse}}$ $x \in \mathbb{R}^N$ is the unique s-sparse soln. to $Az = Ax$".
Consequence : $m \geq \underline{2s}$]

$$\Rightarrow \underline{m \geq 2(2s)} = \underline{c_2 s}.$$

It follows that

$$m \geq c_1 s \ln\left(\frac{N}{m}\right) = c_1 s \ln\left(\frac{eN}{m}\right) - c_1 s$$

$$> \frac{c_1}{2\mu} \ln\left(\frac{eN}{m}\right) - \underline{\frac{c_1}{4} m}$$

$\left(\because \frac{1}{2\mu} < s\right) \longrightarrow$ note: error in recorded lecture here.

Rearranging ,

$$\left(\frac{4+c_1}{4}\right) m > \frac{c_1}{2\mu} \ln\left(\frac{eN}{m}\right)$$

$$m > \frac{2c_1}{4+c_1} \frac{\ln\left(\frac{eN}{m}\right)}{\min\left\{1, \underbrace{c' \frac{\ln\left(\frac{eN}{m}\right)}{m}}\right\}} \geq \boxed{\frac{2c_1}{4+c_1}} \frac{\cancel{\ln\left(\frac{eN}{m}\right)}}{c' \frac{\cancel{\ln\left(\frac{eN}{m}\right)}}{m}}$$

$$\left(c' = \frac{2c_1}{4+c_1}\right)$$

$$= m$$

Contradiction. $\boxtimes$